Gradient, Divergence and Curl

1. Vector differential operator denoted by ∇ (Del / nabla).

$$\nabla = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$$

2. Gradient of scalar point func^n denoted by $(\nabla\phi)$/(grad ϕ)

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

3. Divergence of vector point func^n.

Let $\bar{F} = F_1(x,y,z)\bar{i} + F_2(x,y,z)\bar{j} + F_3(x,y,z)\bar{k}$ then,

$$\nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

4. Curl of vector func^n $\bar{F}$.

Let $\bar{F} = F_1(x,y,z)\bar{i} + F_2(x,y,z)\bar{j} + F_3(x,y,z)\bar{k}$ then

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \text{curl}(\bar{F})$$

Directional Direvative (DD)

$\phi(x,y,z) = c$ at any point $p(x,y,z)$

$\bar{u} = u_1\bar{i} + u_2\bar{j} + u_3\bar{k}$.

- D.D. = $[\nabla\phi]_p \cdot \hat{u}$ where $\hat{u} = \frac{\bar{u}}{|\bar{u}|}$

- scalar quantity

- When direction is along vector $\bar{u} = u_1\bar{i} + u_2\bar{j} + u_3\bar{k}$, vector along t line joining $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ is $\bar{u} = (x_2-x_1)\bar{i} + (y_2-y_1)\bar{j} + (z_2-$

13/3/24

Q. ~~Directio~~ Find the D.D. of $\phi = xy^2 + yz^3$ at $p(1, -1, 1)$ a
$i + 2j + 2\bar{k}$.

Ans. D.D. $= [\nabla\phi]_p \cdot \hat{u}$

$$[\nabla\phi] = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$= y^2\,\bar{i} + (2xy + z^3)\,\bar{j} + 3yz^2\,\bar{k}$$

$$[\nabla\phi]_p = (-1)^2\bar{i} + [(2)(1)(-1) + (1)^3]\,\bar{j} + 3(-1)(1)^2\,\bar{k}$$

$$= \bar{i} - \bar{j} - 3\bar{k}$$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1+4+4}} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

$$\text{D.D.} = [\nabla\phi]_p \cdot \hat{u}$$

$$= \frac{(\bar{i} - \bar{j} - 3\bar{k})(\bar{i} + 2\bar{j} + 2\bar{k})}{3}$$

$$= \frac{1 - 2 - 6}{3} = \frac{-7}{3}$$

Q. Find the D.D. of $\phi = xy + yz + zx$ at $p(1, 0, 1)$ along $\bar{i} + \bar{j} - \bar{k}$.

Ans. D.D. $= [\nabla\phi]_p \cdot \hat{u}$

$$[\nabla\phi] = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$= (y + z)\bar{i} + (x + z)\bar{j} + (y + x)\bar{k}$$

$$[\nabla\phi]_p = (0+1)\bar{i} + (1+1)\bar{j} + (0+1)\bar{k}$$

$$= \bar{i} + 2\bar{j} + \bar{k}$$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{\bar{i} + \bar{j} - \bar{k}}{\sqrt{1+1+1}} = \frac{\bar{i} + \bar{j} - \bar{k}}{\sqrt{3}}$$

13/3/24

$$D.D. = \frac{(\bar{i} + 2\bar{j} + \bar{k})(\bar{i} + \bar{j} - \bar{k})}{\sqrt{3}}$$

$$= \frac{1 + 2 - 1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

Q. Find the D.D. of $\phi = xy + yz^2$ at point $(1, -1, 1)$ in the direction of line PQ where $Q = (2, 1, 2)$.

Ans. $D.D. = [\nabla\phi]_P \cdot \hat{u}$

$$\phi = xy + yz^2$$

$$\bar{u} = (2-1)\bar{i} + (1+1)\bar{j} + (2-1)\bar{k}$$

$$= \bar{i} + 2\bar{j} + \bar{k}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$[\nabla\phi]_P = y\bar{i} + (x + z^2)\bar{j} + (2yz)\bar{k}$$

$$[\nabla\phi]_P = (-1)\bar{i} + (1+1)\bar{j} + [2(-1)(1)]\bar{k}$$

$$= -\bar{i} + 2\bar{j} - 2\bar{k}$$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{\bar{i} + 2\bar{j} + \bar{k}}{\sqrt{1+4+1}} = \frac{\bar{i} + 2\bar{j} + \bar{k}}{\sqrt{6}}$$

$$D.D. = [\nabla\phi]_P \cdot \hat{u} = \frac{(-\bar{i} + 2\bar{j} - 2\bar{k})(\bar{i} + 2\bar{j} + \bar{k})}{\sqrt{6}}$$

$$= \frac{-1 + 4 - 2}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

Q. Find the D.D. of $\phi = 2xz^4 - x^2y$ at the point $(2, -2, 1)$ along $Q(1, -1, 1)$.

Ans. $D.D. = [\nabla\phi]_P \cdot \hat{u}$

13/2/24

D.D. ⇒

$$[\phi\nabla] = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$= (2z^4 - 2xy)\bar{i} + (-x^2)\bar{j} + (8xz^3)\bar{k}$$

$$[\nabla\phi]_P = [2(1)^4 - 2(2)(-2)]\bar{i} + (-(2)^2)\bar{j} + [8(2)(1)^3]\bar{k}$$

$$= 10\bar{i} - 4\bar{j} + 16\bar{k}$$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{-\bar{i} + \bar{j}}{\sqrt{1+1}} = \frac{-i+j}{\sqrt{2}}.$$

$$D.D. = [\nabla\phi]_P \cdot \hat{u}$$

$$= \frac{(10\bar{i} - 4\bar{j} + 16\bar{k})(-i+j)}{\sqrt{2}}$$

$$= \frac{-10-4}{\sqrt{2}} = -\frac{14}{\sqrt{2}} = -7\sqrt{2}$$

Solenoid Vector Field.

If $\nabla\bar{F} = 0$, $\bar{F}$ is said to be solenoidal

Irrotational vector field and scalar potential

If curl $\bar{F} = 0 = \nabla\times\bar{F}$, then F is said to be irrotational.

If $\bar{F}$ is irrotational, then there exists a scalar func^n such that $\bar{F} = \nabla\phi$, ϕ is called scalar potential of $\bar{F}$.

To find scalar potential $\phi(x,y,z)$

$\because \bar{F} = \nabla\phi$, $d\phi = \bar{F}\cdot d\bar{r}$

$\therefore d\phi = \nabla\phi\cdot d\bar{r} = \bar{F}\cdot d\bar{r}$

$$= (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

$$\nabla\phi = \int_{y,z\text{-const.}} F_1\,dx + \int_{\substack{x\text{-free}\\ z\text{-const}}} F_2\,dy + \int_{x,y\text{-free}} F_3\,dz$$

13/3/24

$d\phi = F_1\,dx + F_2\,dy + F_3\,dz \longrightarrow$ exact D.E.

$$\phi = \int\limits_{y,z\ \text{const}} F_1\,dx + \int\limits_{x,z\ \text{const}} F_2\,dy + \int\limits_{x,y\ \text{const}} F_3\,dz$$

14/3/24

Q. Show that $\bar{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is irrotational. Find the scalar potential ϕ such that $\bar{F} = \nabla\phi$.

Ans. Given

$\bar{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

$F_1 = x^2-yz$

$F_2 = y^2-zx$

$F_3 = z^2-xy$

$$\nabla\times\bar{F} = \text{Curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial(z^2-xy)}{\partial y} - \frac{\partial(y^2-zx)}{\partial z}\right] - \bar{j}\left[\frac{\partial(z^2-xy)}{\partial x} - \frac{\partial(x^2-yz)}{\partial z}\right] + \bar{k}\left[\frac{\partial(y^2-zx)}{\partial x} - \frac{\partial(x^2-yz)}{\partial y}\right]$$

$$= (-x+x)\bar{i} - (-y+y)\bar{j} + (-z+z)\bar{k}$$

$$= 0$$

$\therefore$ $\bar{F}$ is irrotational

$$\phi = \int_{y,z\text{-const}} F_1\,dx + \int_{\substack{x\text{-free}\\ z\text{-const}}} F_2\,dy + \int_{x,y\text{-free}} F_3\,dz$$

$$= \int_{z,y\text{-const}} x^2 - yz\,dx + \int_{\substack{x\text{-free}\\ z\text{-const}}} y^2 - zx\,dy + \int_{x,y\text{-free}} z^2 - xy\,dz$$

$$\cancel{= c(2x)x}$$

$$\cancel{= \frac{x^3}{3} + \frac{y^3}{3} - zy}$$

$$= \frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + C$$

$$= \frac{x^3 + y^3 - z^3 - 3xyz}{3} + C$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$= (x^2 - yz)\bar{i} + (-xz + y^2)\bar{j} + (-xy - z^2)\bar{k}$$

$$= (x^2 - yz)\bar{i} + (y^2 - xz)\bar{j} + (-z^2 - xy)\bar{k}$$

$$= \bar{F}$$

Hence Proved.

Q. Show that $\bar{F} = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$ is irrotational. Find scalar potential such that $\bar{F} = \nabla\phi$.

Ans. Given

$\bar{F} = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$

$F_1 = 6xy + z^3$

$F_2 = 3x^2 - z$

$F_3 = 3xz^2 - y$

$$\nabla \times F = \text{Curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz-y \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial(3xz-y)}{\partial y} - \frac{\partial(3x^2-z)}{\partial z}\right] - \bar{j}\left[\frac{\partial(3xz^2-y)}{\partial x} - \frac{\partial(6xy+z^3)}{\partial z}\right]$$

$$+ \bar{k}\left[\frac{\partial(3x^2-z)}{\partial x} - \frac{\partial(6xy+z^3)}{\partial y}\right]$$

$$= [(-1)+1]\,\bar{i} - \bar{j}\,[3z^2 - 3z^2] + \bar{k}\,[6x - 6x]$$

$$= (3z^2 - 3z^2)\,\bar{j}$$

$$= \cancel{3z(z-1)\bar{j} \neq 0} = 0$$

SharkCoders

$\therefore$ $\bar{F}$ is irrotational.

$$\phi = \int_{y,z\text{-const}} F_1\,dx + \int_{\substack{x\text{-free}\\ z\text{-const}}} F_2\,dy + \int_{\cancel{x,y\text{-free}}} F_3\,dz$$

$$= \int (6xy + z^3)\,dx + \int (3x^2 - z)\,dy + \int (3xz^2 - y)\,dz$$

$$= \frac{6x^2}{2}(y) + xz^3 + (-zy) + 0$$

$$= 3x^2y + xz^3 - yz + C$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$= (6x + z^3)\,\bar{i} + (3x^2 - z)\,\bar{j} + (3xz^2 - y)\,\bar{k}$$

$$= \bar{F}$$

Hence Proved

Q. Show that $\bar{F} = \cancel{(6xy+z^3)\bar{i} + (3x^2-z)\bar{j} + (3xz^2-y)\bar{k}}$ $(y\sin z - \sin x)\bar{i} + (x\sin z + 2yz)\bar{j} + (xy\cos z + y^2)\bar{k}$ is irrotational.

Find scalar potential such that $\bar{F} = \nabla\phi$.

Ans. Curl $\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$

$$= i\left[\frac{\partial(xy\cos z + y^2)}{\partial y} - \frac{\partial(x\sin z + 2yz)}{\partial z}\right] - \bar{j}\left[\frac{\partial(xy\cos z + y^2)}{\partial x} - \frac{\partial(y\sin z - \sin x)}{\partial z}\right]$$

$$+ \bar{k}\left[\frac{\partial(x\sin z + 2yz)}{\partial x} - \frac{\partial(y\sin z - \sin x)}{\partial y}\right]$$

$$= (x\cos z + 2y - x\cos z - 2y)\bar{i} - (y\cos z - y\cos z)\bar{j} + (\sin z - \sin z)\bar{k}$$

$$= 0$$

SharkCoders

$\therefore \bar{F}$ is irrotational

$$\phi = \int\limits_{y,z-\text{const.}} F_1\, dx + \int\limits_{\substack{x,-\text{free} \\ z-\text{const.}}} F_2\, dy + \int\limits_{x,y-\text{const.}} F_3\, dz$$

$$= \int (y\sin z - \sin x)\,dx + \int\limits_{x,-\text{free}} (x\sin z + 2yz)\,dy + \int (xy\cos z + y^2)\,dz$$

$$= (xy\sin z + \cos x) + (z^2 y) + (0) + c$$

$$= xy\sin z + \cos x + yz^2 + c$$

Hence Proved.

15/3/24

Q1. Evaluate $\int_0^1 \int_{x^2}^{2-x} y \, dy \, dx$

Ans. $I = \int_0^1 \int_{x^2}^{2-x} y \, dy \, dx$

$= \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$

$= \int_0^1 \left[\frac{4+x^2-4x}{2} - \frac{x^4}{2}\right] dx$

$= \int_0^1 \left[2 + \frac{x^2}{2} - 2x - \frac{x^4}{2}\right] dx$

$= \left[2x + \frac{x^3}{6} - \frac{2x^2}{2} - \frac{x^5}{10}\right]_0^1$

$= \left(2(1) + \frac{(1)^3}{6} - (1)^2 - \frac{(1)^5}{10}\right)$

$= 2 + \frac{1}{6} - 1 - \frac{1}{10}$

$= \frac{60+5-30-3}{30} = \frac{65-33}{30} = \frac{32}{30} = \frac{16}{15}$

Q2. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

Ans. $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

$= \int_0^1 \left[\frac{xy^2}{2}\right]_{x^2}^{2-x} dx$

$= \int_0^1 \left[\left(\frac{x(4+x^2-4x)}{2}\right) - \left(\frac{x(x)^4}{2}\right)\right] dx$

$= \int_0^1 \left[\frac{4x+x^3-4x^2-x^5}{2}\right] dx$

$= \frac{1}{2}\int_0^1 (4x+x^3-4x^2-x^5) \, dx$

$$= \frac{1}{2}\left[2x^2 + \frac{x^4}{4} - \frac{4x^3}{3} - \frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{2}\left[2(1)^2 + \frac{(1)^4}{4} - \frac{4(1)^3}{3} - \frac{(1)^6}{6}\right]$$

$$= \frac{1}{2}\left(2 + \frac{1}{4} - \frac{4}{3} - \frac{1}{6}\right)$$

$$= \frac{3}{8}$$

Q3. Evaluate $\int_{-2}^{1}\int_{x^2}^{2-x} y \, dx \, dy$

Ans. $\int_{-2}^{1}\left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx = I.$

$$I = \int_{-2}^{1}\left[\left(\frac{4 + x^2 - 4x}{2}\right) - \left(\frac{x^4}{2}\right)\right] dx$$

$$= \cancel{\int_{-2}^{1}\left(2 + \frac{x^2}{2} - 2x - \frac{x^4}{2}\right) dx}$$

$$= \int_{-2}^{1}\left(\frac{4 + x^2 - 4x - x^4}{2}\right) dx$$

$$= \frac{1}{2}\left[4x + \frac{x^3}{3} - \frac{4x^2}{2} - \frac{x^5}{5}\right]_{-2}^{1}$$

$$= \frac{1}{2}\left[\left(4(1) + \frac{(1)^3}{3} - 2(1)^2 - \frac{(1)^5}{5}\right) - \left(4(-2) + \frac{(-2)^3}{3} - 2(-2)^2 - \frac{(-2)^5}{5}\right)\right]$$

$$= \frac{1}{2}\left[\left(4 + \frac{1}{3} - 2 - \frac{1}{5}\right) - \left(-8 - \frac{8}{3} - 8 + \frac{32}{5}\right)\right]$$

$$= \frac{36}{5}$$

Date
Page

Q4. Evaluate $\int_0^{2\pi}\int_{r=0}^{1} (1+r^2)\, dr\, d\theta$

Ans $I = \int_0^{2\pi}\int_{r=0}^{1} (1+r^2)\, dr\, d\theta$

$= \left[\int_0^{2\pi} d\theta\right]\left[\int_{r=0}^{1} (1+r^2)\, dr\right]$

$= [\theta]_0^{2\pi} \cdot \left[r + \frac{r^3}{3}\right]_0^1$

$= (2\pi)\left[\left(1+\frac{1}{3}\right) - 0\right]$

$= 2\pi\left(\frac{4}{3}\right) = \frac{8\pi}{3}$

Q5. Evaluate $\int_0^{\pi/2}\int_{r=0}^{a} r\sin\theta\, dr\, d\theta$

Ans. $I = \int_0^{\pi/2}\int_{r=0}^{a} r\sin\theta\, dr\, d\theta$

$= \left[\int_0^{\pi/2} d\theta\right]\left[\int_{r=0}^{a} r\sin\theta\, dr\right]$

$= \cancel{[\theta]_0^{\pi/2} \cdot \left[\frac{r^2 \sin\theta}{2}\right]} \left[\frac{r^2}{2}\right]$

$= [\theta]_0^{\pi/2} \cdot \sin\theta\left[\frac{r^2}{2}\right]_0^a$

$= (\pi/2) \cdot \int_0^{\pi/2} \sin\theta\left(\frac{a^2}{2}\right) d\theta$

$= \frac{a^2}{2}\left[-\cos\theta\right]_0^{\pi/2} = \frac{a^2}{2}\left[-(0) - (-1)\right] = \frac{a^2}{2}$

$\int_0^{\pi/2}\left[\frac{r^2}{2}\sin\theta\right]_0^a d\theta$

$\int_0^{\pi/2} \frac{a^2}{2}\sin\theta$

$\frac{a^2}{2}\left[-\cos\theta\right]_0^{\pi/2}$

$\frac{a^2}{2}(0+1)$

$\frac{a^2}{2}$

Q6. Evaluate $\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-(x^2+y^2)} \cdot dx\, dy$.

Ans. $I = \int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$

$$\cancel{= \int_{y=0}^{\infty} \left[\frac{e^{-(x^2+y^2)}}{2x}\right]_0^{\infty} dy}$$

$$\cancel{= \int_{y=0}^{\infty} \left[\left(\frac{e^{-\infty+y^2}}{2\infty}\right) - e^{-}\right.}$$

~~Let~~ $\cancel{e^{(x^2+y^2)}} =$

~~Let~~ e

$$I = \int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-x^2} \cdot e^{-y^2}\, dx\, dy$$

$$= \int_{x=0}^{\infty} \left[\frac{e^{-x^2} \cdot e^{-y^2}}{-2y}\right]_0^{\infty} dx$$

$$I = \int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-x^2} \cdot e^{-y^2}\, dx\, dy$$

$$= \int_{x=0}^{\infty} e^{-x^2}\, dx \cdot \int_{y=0}^{\infty} e^{-y^2}\, dy$$

$x^2 = t$ $\qquad\qquad$ $y^2 = u$

$2x\, dx = dt$ $\qquad\qquad$ $2y\, dy = du$

$dx = \dfrac{dt}{2x}$ $\qquad\qquad$ $dy = \dfrac{du}{2y}$

x	$= 0$	∞
t	0	∞

y		
u		

Date ______
Page ______

Q6. $I = \underbrace{\int_0^{\infty} e^{-x^2}\,dx}_{I_1} \cdot \underbrace{\int_0^{\infty} e^{-y^2}\,dy}_{I_2}$

Ans. $-x^2 = -t$ $\qquad$ $-y^2 = -u^2$

$x^2 = t$ $\qquad$ $y^2 = u$

$x = t^{1/2}$ $\qquad$ $y = u^{1/2}$

$\cancel{\int_0^{\infty} e^{-(t^{1/2})^2}\,dx} \cancel{\int_0^{\infty} e^{(u^{1/2})^2}\,dy} = \cancel{f}$

$dx = \frac{1}{2} t^{-1/2}\,dt$

$I_1 = \int_0^{\infty} e^{-t} \frac{1}{2} t^{-1/2}\,dt$

$= \frac{1}{2}\int_0^{\infty} e^{-t}\, t^{-1/2}\,dt$

$= \frac{1}{2}\,\Gamma\left(-\frac{1}{2}+1\right)$

$= \frac{1}{2}\,\Gamma\frac{1}{2}$

$= \frac{1}{2}\sqrt{\pi}$

$I = I_1 \cdot I_2$

$= \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2}$

$= \frac{\pi}{2}$

$I_2 = \int_0$ $\qquad dy = \frac{1}{2} u^{-1/2}\,du$

$I_2 = \int_0^{\infty} e^{-u} \cdot \frac{1}{2} \cdot u^{-1/2}\,du$

$= \frac{1}{2}\int_0^{\infty} e^{-u} \cdot u^{-1/2}\,du$

$= \frac{1}{2}\,\Gamma\left(-\frac{1}{2}+1\right) = \frac{1}{2}\,\Gamma\frac{1}{2}$

$= \frac{\sqrt{\pi}}{2}$

Q7. Evaluate $\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} x^2y^2\,dx\,dy$

Ans. $I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} x^2y^2\,dx\,dy$

$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} x^2y^2\,dy\,dx$

~~Using polar coordinates,~~

$= \int_{x=0}^{1}\left[\frac{x^2y^3}{3}\right]_0^{\sqrt{1-x^2}} dx = \int_{x=0}^{1}\frac{x^2(1-x^2)\sqrt{1-x^2}}{3}\,dx$

$= \frac{1}{3}\int_{x=0}^{1}(x^2-x^4)(1-x^2)^{1/2}$

$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} x^2y^2\,dy\,dx$

$y = 0 \qquad y = \sqrt{1-x^2}$

$y^2 = 1-x^2$

$\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\frac{1}{2} \qquad x^2+y^2 = 1 = x^2 = r^2$

$\Rightarrow r = 1$

$x = 0 \qquad \frac{1}{2}\sqrt{\pi} \qquad x = 1$

$\frac{3}{2}\Gamma\frac{3}{2} \quad \frac{3}{2}\times\frac{1}{2}\Gamma\frac{1}{2} \quad x^2+y^2 = 1$ at $C\in(1,0)$ $r = 1$

$\frac{3}{4}\sqrt{\pi}$

$4\times3\times2$ $\frac{\pi}{16\times6}$ $\frac{\pi}{96}$

$\int_0^1\int_0^{\sqrt{1-x^2}} x^2y^2\,dx\,dy$

$\int_0^1\left[\frac{x^2y^3}{3}\right]_0^{\sqrt{1-x^2}} dx$

$\int_0^1 x^2(1-x^2)\sqrt{1-x^2}\,dx$

$\int_0^1 x^2(1-x^2)^{3/2}\,dx$

Put $x^2 = t \Rightarrow x = t^{1/2}$

$dt = 2x\,dx \Rightarrow \frac{dt}{2x} = dx$

$dx = \frac{1}{2}t^{-1/2}\,dt$

$= \frac{1}{3}\int_0^1 t(1-t)^{3/2}\,\frac{1}{2}(t^{-1/2})\,dt$

$\frac{1}{6}\int_0^1 t^{1/2}(1-t)^{3/2}$

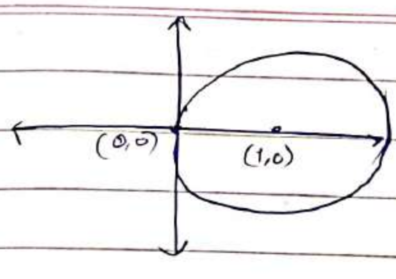
$\frac{1}{6}\beta\left(\frac{3}{2},\frac{5}{2}\right)$

$\frac{1}{6}\frac{\Gamma\frac{3}{2}\,\Gamma\frac{5}{2}}{\Gamma(3+1)}$

$\frac{1}{6}\left(\frac{1}{2}\right)$.

$\frac{1}{6}\times\frac{1}{2}\sqrt{\pi}\times\frac{3}{4}\sqrt{\pi}$

$\frac{\pi}{16}\times\frac{1}{2}$



$x^2 + y^2 = 1$

$x^2(1 - x^2) = x^2 - x^4$

$dx\, dy = r\, dr\, d\theta$

$r : 0$ to 1

$\theta : -\pi/2$ to $\pi/2$

$$I = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{1} x^2 - x^4 \, r\, dr\, d\theta$$

$$I = \frac{1}{6} B\left(3/2, 5/2\right)$$

$$= \frac{1}{6} \frac{\Gamma 3/2 \cdot \Gamma 5/2}{\Gamma 4}$$

$$= \frac{1}{6} \frac{\Gamma 1/2 + 1 \quad \Gamma \frac{3}{2} + 1}{24}$$

$$= \left(\frac{1}{6}\right) \frac{1/2 \, \Gamma 1/2 \cdot 3/2 \, \Gamma 3/2}{3!} \quad -\frac{1}{2} + 1$$

$$= \frac{1/2 \sqrt{\pi} \cdot 3/2 \cdot 1/2 \, \Gamma 1/2}{36}$$

$$= \left(\frac{3}{8}\pi\right)\left(\frac{1}{36}\right) = \frac{\pi}{12 \times 8} = \frac{\pi}{96}.$$

Q8. $\iint_R x^2y^2 \, dx\, dy$ where R is region $x^2+y^2=1$ first quadrant.

Ans $I = \iint_R x^2y^2 \, dx\, dy$

$x^2(1-x^2)$.

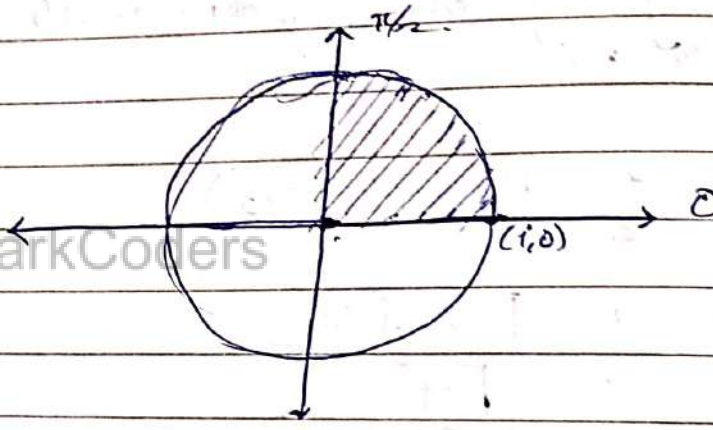
$x = r\cos\theta$

$y = r\sin\theta$

$dx\, dy = r\, dr\, d\theta$

$x^2+y^2 = r^2$

$r = 1$



r : 0 to 1

θ : 0 to $\pi/2$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} \cancel{x^2 - x^4\,dx}\; r^2\cos^2\theta \cdot r^2\sin^2\theta \; r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{r} r^5 \sin^2\theta\cos^2\theta \, dr\, d\theta$$

$$\cancel{= \int_{\theta=0}^{\pi/2} d\theta \sin^2\theta\cos^2\theta\, d\theta}$$

$$= \left[\int_{\theta=0}^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta\right]\left[\int_{r=0}^{1} r^5\, dr\right]$$

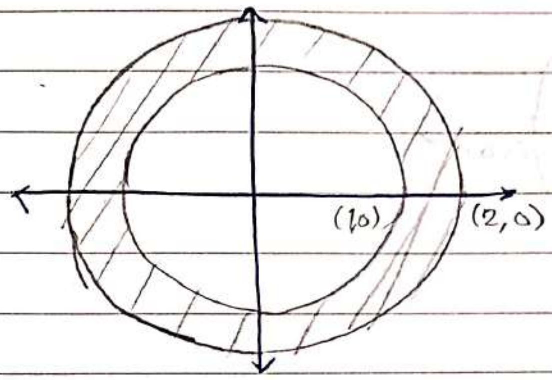
$$= \left[\frac{(1)\cdot(1)}{(4)(2)}\right]\left(\frac{\pi}{2}\right)\cdot\left[\frac{r^6}{6}\right]_0^1$$

$$= \left(\frac{\pi}{16}\right)\left(\frac{1}{6}\right) = \frac{\pi}{96}$$

Q.9. $\iint_R \frac{1}{x^2+y^2}\,dx\,dy$ where R is region bounded by circle $x^2+y^2=1$ and $x^2+y^2=4$.

Ans. $I = \iint_R \frac{1}{x^2+y^2}\,dx\,dy$

$R: \; x^2+y^2=1$ and $x^2+y^2=4$



$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r: 1, 2$

$\theta: 0$ to 2π

$$I = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} \frac{1}{r^2}\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=1}^{2} \frac{1}{r}\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} d\theta \cdot [\log r]_1^2 = (2\pi)\cdot[\log(2)-\log(1)]$$

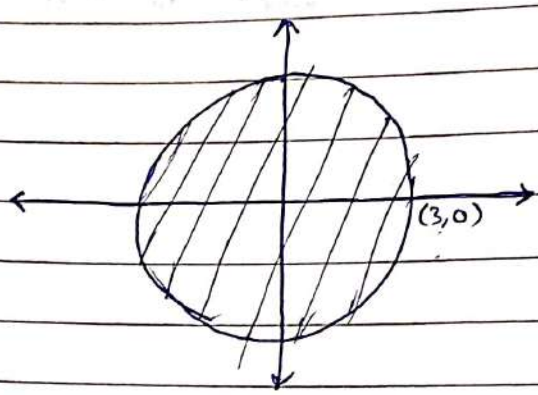
$$= 2\pi \log 2$$

Q10. Evaluate $\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$ where R is circle $x^2+y^2=9$

Ans. $I = \iint_R \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$

$R: \quad x^2+y^2 = 9 = r^2$

$r = 3$



$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r: \quad 0, \text{ to } 3$

$\theta: \quad 0 \text{ to } 2\pi$

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} \frac{1}{r}\, r\,dr\,d\theta$$

$$= \left[\int_{\theta=0}^{2\pi} d\theta\right]\left[\int_{r=0}^{3} dr\right] = (2\pi)\cdot(3) = 6\pi$$

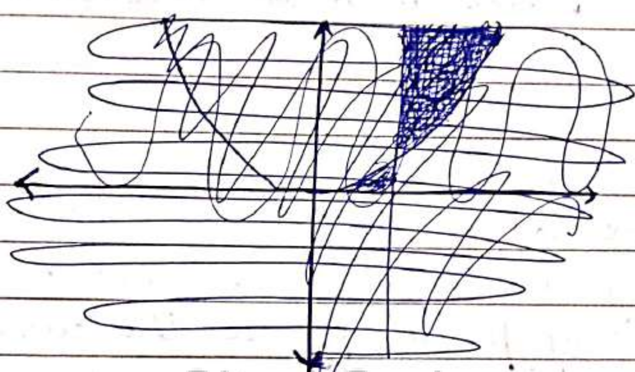
classmate
Date
Page

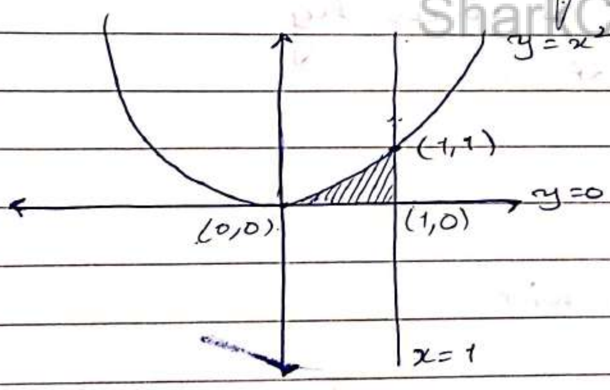
Q11. Evaluate $\iint_R xy^2 \, dx \, dy$ where R is region bounded by $y = x^2$, $y = 0$, $x = 1$.

Ans $I = \iint_R xy^2 \, dx \, dy$

$R = y = x^2 \quad y = 0 \quad x = 1 \quad y = 1, \; x = 1$

$x = 0, \; y = 0$





$y = x^2$

$x = 1$

$y = 1$

$x = 0$ to 1

$y = 0$ to 1

$\cancel{x = r\cos\theta}$

$\cancel{y = r\sin\theta}$

$\cancel{dx\,dy = r\,dr\,d\theta}$

$\cancel{xy^2 = r\cos\theta \, r^2\sin^2\theta}$

$\cancel{= r^3\cos\theta(1-\cos^2\theta)}$

Consider strip $\parallel$ y (int. y first),

$$I = \int_{x=0}^{1}\int_{y=0}^{1} xy^2 \, dy \, dx$$

$$= \int_{x=0}^{1} \left[\frac{xy^3}{3}\right]_0^1 dx$$

$$= \int_{x=0}^{1} \frac{x}{3} \, dx$$

$$= \left[\frac{x^2}{6}\right]_0^1 = \frac{1}{6}$$

VECTOR CALCULUS

~~Line Integral~~

Line Integral

Work Done by a Force

Conservation Field

Vector Calculus

Green's Theorem

Stoke's Theorem

Gauss-Divergence Theorem

Line Integral

line integral = area of surface area

Work Done By A Force

Work done by force $\bar{F} = f_1(x,y,z)\bar{i} + F_2(x,y,z)\bar{j} + f_3(x,y,z)\bar{k}$

$$= \int_C \bar{F}\, d\bar{r}$$

$$= \int_{C:A}^{B} (F_1\, dx + F_2\, dy + F_3\, dz)$$

Method I: Curve C is known.

$$\int_C \bar{F}\, d\bar{r} = \int_{x=a}^{b} \varphi(x)\, dx$$

Method II: Curve C is known in parametric form.
(~~eg~~ line, circle, and ellipse)

$$\int_C \bar{F}\, d\bar{r} = \int_{t=a}^{b} \varphi(t)\, dt$$

Straight line:

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t \quad \text{(say)}$$

Circle: $(x^2+y^2=a^2)$

$x = a\cos t$

$y = a\sin t$

$z = 0$

Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right)$

$x = a\cos t$

$y = b\sin t$

$z = 0$

$\boxed{\text{Conservative Field}}$

(irrotational) - $\int_C \bar{F}\, dr = 0$. (Curl $\bar{F}$)

$\boxed{\text{Green's Theorem}}$

$$\oint \bar{F}\, dr = \oint_C u\,dx + v\,dy$$

$$= \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$$

SharkCoders

Stoke's Theorem

$$\oint_C \bar{F}\, dr = \iint_S (\nabla \times \bar{F})\, d\bar{s}$$

$$= \iint_S (\nabla \times \bar{F})\, \hat{n}\, ds$$

$\therefore d\bar{s} = \hat{n}\, ds$ where $\hat{n}$ — outer normal unit vector to surface S in anti-clockwise direction.

Gauss - Divergence Theorem

$$\iint \bar{F} \cdot d\bar{s} = \iiint_V \nabla \cdot \bar{F}\, dv$$

2/4/24

Unit IV: Vector Calculus

VECTOR INTEGRATION

Line Integral

Line integral = area of curved surface

- any integral which is to be evaluated along a curve ~~called~~ ~~is~~

- mathematically,

$\bar{F}(x,y,z)$ be a vector point function define in space C be any smooth curve and $\bar{r}$ be a position vector of any point P on the curve C, then the line integral $\bar{F}$ over C is given by

Line integral = $\int_C \bar{F}\, d\bar{r}$

SharkCoders

Applications

1. Evaluate the mass of wire

2. Helps to calculate ~~mo~~ moment of inertia, centre of mass of wire

3. They help to evaluate work ~~done~~. done.

~~Also~~

2/4/24

classmate
Date
Page

Work Done by a force

If $\bar{F} = f_1(x,y,z)\hat{i} + f_2(x,y,z)\hat{j} + f_3(x,y,z)\hat{k}$ is a force acting on a particle ~~which~~ which is moving along a curve C from point A to point B. The ~~work by~~ $\bar{F}$ work done by $\bar{F}$ ~~during the displacement~~ during the displacement $d\bar{r}$ is $\bar{F}\, d\bar{r}$.

- Work done $= \int^{B} \bar{F}\, d\bar{r}$

$$= \int_{C:A}^{B} (f_1 dx + f_2 dy + f_3 dz)$$

Working Rule

Method I: If ~~eq~~ eq^n ~~eq~~ of the curve C are known the line integral is usually computed by ~~expres~~ expressing in terms ~~of~~ of single variable.

- Work done $= \int_{C:A}^{B} \bar{F}\cdot d\bar{r}$

$$= \int_{C:A}^{B} (f_1\, dx + f_2\, dy + f_3\, dz)$$

- $$\int_C \bar{F}\cdot d\bar{r} = \int_{x=a}^{b} \varphi(x)\,dx$$

Method II: If eq^n of curve C are known in parame-ic form (e.g.: line, circle, and ellipse), then

$$\int_C \bar{F}\, d\bar{r} = \int_{t=a}^{b} \varphi(t)\, dt$$

2/4/24

Unit IV: Vector Calculus

VECTOR INTEGRATION

Line Integral

Line integral = area of curved surface

- any integral which is to be evaluated along a curve ~~called~~ ~~to~~

- mathematically,
  $\bar{F}(x,y,z)$ be a vector point function define in space, C be any smooth curve and $\bar{r}$ be a position vector of any point P on the curve C, then the line integral $\bar{F}$ over C is given by
  Line integral = $\int_C \bar{F} \, d\bar{r}$

Applications

1. Evaluate the mass of wire

2. Helps to calculate ~~mo~~ moment of inertia, centre of mass of wire

3. They help to evaluate work ~~done~~. done.

Note:

2/4/24

classmate
Date
Page

Work Done by a Force

If $\bar{F} = f_1(x,y,z)\,\bar{i} + f_2(x,y,z)\,\bar{j} + f_3(x,y,z)\,\bar{k}$ is a force acting on a particle ~~which~~ which is moving along a curve C from point A to point B. The ~~the~~ ~~by F~~ work done by $\bar{F}$, ~~during the displacement~~ during the displacement $d\bar{r}$ is $\bar{F}\,d\bar{r}$.

$$\cdot\ \text{Work done} = \int^{B} \bar{F}\, d\bar{r}$$

$$= \int_{C:A}^{B} (f_1\,dx + f_2\,dy + f_3\,dz)$$

Working Rule

Method I: If ~~eq~~ eq^n ~~of~~ of the curve C are known the line integral is usually computed by ~~express~~ expressing in terms ~~of~~ of single variable.

$$\cdot\ \text{Work done} = \int_{C:A}^{B} \bar{F}\cdot d\bar{r}$$

$$= \int_{C:A}^{B} (F_1\,dx + F_2\,dy + f_3\,dz)$$

$$\cdot\ \int_C \bar{F}\cdot d\bar{r} = \int_{x=a}^{b} \varphi(x)\,dx$$

Method II: If eq^n of curve C are known in parametric form (e.g.: line, circle, and ellipse), then

$$\int_C \bar{F}\, d\bar{r} = \int_{t=a}^{b} \varphi(t)\,dt$$

2/4/24

Parametric eq^n of sum standard curve

1) Eq^n of straight line A(x_1, y_1, z_1) & B(x_2, y_2, z_2). If straight line joining A and B, then its eq^n is given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t \text{ (say)}$$

$x - x_1 = t(x_2 - x_1)$

~~$x - x_1 =$~~

$x = x_1 + t(x_2 - x_1)$

$y = t(y_2 - y_1) + y_1$

$z = t(z_2 - z_1) + z_1$

2) Eq^n of circle

Let eq^n of circle be $x^2 + y^2 = a^2$, $z = 0$

The parametric eq^n of circle are as follows

$x = a\cos t$

$y = a\sin t$

$z = 0$

3) Eq^n of ellipse

Let the eq^n of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $z = 0$

The parametric eq^{n}s of ellipse are as follows:

$x = a\cos t$

$y = b\sin t$

$z = 0$

2/4/24

Conservative field (irrotational)

$$\int_C \bar{F}\, d\bar{r} = 0$$

Q. Evaluate $\int_C \bar{F}\, d\bar{r}$. where $\bar{F} = 3x^2\,\bar{i} + (2xz - y)\,\bar{j} + z\,\bar{k}$.
along the curve $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t = 0$ to $t = 1$.

Ans $\bar{F} = 3x^2\,\bar{i} + (2xz - y)\,\bar{j} + z\,\bar{k}$

$C : x = 2t^2, y = t, z = 4t^2 - t$

$$\text{Line integral} = \int_C \bar{F}\, d\bar{r}$$

$$= \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$

Now along the curve $x = 2t^2$.

$dx = 4t\, dt$

Similarly.

$y = t$ $\qquad$ $z = 4t^2 - t$

$dy = dt$ $\qquad$ $dz = (8t - 1)\, dt$

$$\text{Line integral} = \int \left[(3x^2)\, dx + (2xz - y)\, dy + (z)\, dz \right] \quad (1)$$

$$= \int_C \left[3(2t^2)^2 (4t\, dt) + [(2)(2t^2)(4t^2 - t) - (t)](dt) + (4t^2 - t)(8t - 1)\, dt \right]$$

$16t^4 - 4t^3 - t$ $\qquad$ $24t^2 - 3t$

$$= \int_C \left[48t^4\, dt + (4t^2(4t^2 - t) - t)\, dt + (32t^2 - 4t - 8t^2 + t)\, dt \right]$$

2/4/24

$$= \int_C \left[48t^5\,dt + (16t^4 - 4t^3 - t)\,dt + (24t^2 - 3t)\,dt \right]$$

$$\cancel{= \int_{t=0}^{1} \left[(64t^4 - 4t^3 + 24t^2 - 4t)\,dt \right]}$$

$$\cancel{= \left[\frac{64t^5}{5} - \frac{4t^4}{4} + \frac{24t^3}{3} - \frac{4t^2}{2} \right]_0^1}$$

$$\cancel{= \left[\frac{64t^5}{5} - t^4 + 8t^3 - 2t^2 \right]_0^1}$$

$$= \left[\frac{64}{5} - 1 + 8 - 2 \right] \cancel{= 17.8}$$

$\frac{71}{5}$

$$= \int_{t=0}^{1} \left[\left(48t^5 + 16t^4 - 4t^3 + 24t^2 - 4t^2 \right) dt \right.$$

$$= \left[\frac{48t^6}{6} + \frac{16t^5}{5} - \frac{4t^4}{4} + \frac{24t^3}{3} - \frac{4t^3}{3} \right]_0^1$$

$$= \left[8t^6 + \frac{16t^5}{5} - t^4 + 8t^3 - \frac{4t^3}{3} \right]_0^1$$

$$= \left[8 + \frac{16}{5} - 1 + 8 - \frac{4}{3} \right] = \frac{256}{15} \; ?$$

2/4/24

Q. $\bar{F} = x\,\bar{i} + 2xy\,\bar{j} + z\bar{k}$;

$x = 2t,\ y = t;\ z = 1.$; $t = 0$ to $t = 1$

Line integral $= \int_C \bar{F}\, d\bar{r}$

$= \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$.

$= \int_C x\, dx + 2xy\, dy + z\, dz.$

$x = 2t \qquad y = t \qquad z = 1$

$dx = 2\,dt \qquad dy = dt \qquad dz = 0$

Line integral $= \int_{t=0}^{1} \left[(2t)(2\,dt) + (2)(2t)(t)(dt) + (1)(0) \right]$

$= \int_{t=0}^{1} \left[(4t + 4t^2) \right] dt$

$= \left[\frac{4t^2}{2} + \frac{4t^3}{3} \right]_0^1 = \left[\frac{4}{2} + \frac{4}{3} \right] = 3.34$

3/4/24

Q. $\bar{F} = (2xz^3 + 6y)\hat{i} + (6x - 2yz)\hat{j} + (3x^2z^2 - y^2)\bar{k}$.
$\int_C \bar{F}\,d\bar{r}$ where C is a straight line joining (0,0,0), (1,1,1).

Ans. Line integral $= \int_C \bar{F}\,d\bar{r}$

$$= \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \int_C \left[(2xz^3 + 6y)\,dx + (6x - 2yz)\,dy + (3x^2z^2 - y^2)\,dz\right]$$

Eq^n of straight line;

$(x_1, y_1, z_1) = (0,0,0)$

$(x_2, y_2, z_2) = (1,1,1)$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \text{ (say)}$$

$$x = y = z = t$$

$\therefore x = t \qquad dx = dt$

$y = t \qquad dy = dt$

$z = t \qquad dz = dt$

Line integral $= \int_{t=0}^{1} [(2)(t)(t)^3 + 6t]\,dt + (6t - 2t^2)\,dt + (3t^4 - t^2)\,dt$

$$= \int_{t=0}^{1} (\cancel{2t^4} + 6t + 6t - 2t^2 + \cancel{3t^4} - \cancel{t^2})\,dt$$

$$= \int_{t=0}^{1} (5t^4 - 3t^2 + 12t)\,dt$$

3/4/24

$$= \iiint_{z=0}^{1} \left[t^5 - t^3 + 6t^2 \right]_0^1$$

$$= (1 - 1 + 6) = 6$$

Q. ~~Evaluate the~~ for $\bar{F} = 2xy\,\bar{i} + (x^2-1)\,\bar{j} + (yz)\,\bar{k}$. $\int$ ~~for dr~~ along the straight line joining (0,0,0) to (1,2,1).

Ans. Line integral $= \int_C \bar{F}\, d\bar{r}$

$$= \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \int_C [(2xy)\,dx + (x^2-1)\,dy + (yz)\,dz]$$

Eq^n of straight line,

$(x_1, y_1, z_1) = (0,0,0)$

$(x_2, y_2, z_2) = (1,2,1)$

$$\frac{x-0}{1-0} = \frac{y-0}{2-0} = \frac{z-0}{1-0} = t$$

$\therefore x = t \qquad dx = dt$

$y = 2t \qquad dy = 2\,dt$

$z = t \qquad dz = dt$

$y = 2t$.

$0 = 2t \Rightarrow t = 0$.

$2 = 2t \Rightarrow t = 1$.

Line integral $= \int_{t=0}^{1} (2(2t^2))\,dt + (t^2-1)\,2dt + (2t^2)\,dt$

$$= \int_{t=0}^{1} \cancel{4t^2} + 2(t^2-1)$$

$$= \int_{t=0}^{1} (4t^2 + 2t^2 - 2 + 2t^2)\,dt$$

$$= \int_{0}^{1} 8t^2 - 2\,dt$$

$$= \left[\frac{8t^3}{3} - 2t\right]_0^1$$

$$= \frac{8}{3} - 2 = 0.67$$

Q. Find the work done by $\bar{F} = (x^2)\bar{i} + (2xy)\bar{j} + (z)\bar{k}$ in moving a particle along the straight line joining $(1,0,2)$ to $(3,1,1)$.

Ans. Line integral $= \int \bar{F}\,d\bar{r}$

$$= \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \int_C [(x^2)\,dx + (2xy)\,dy + (z)\,dz]$$

Eqn of straight line,

$(x_1, y_1, z_1) = (1, 0, 2)$

$(x_2, y_2, z_2) = (3, 1, 1)$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

$$\frac{x - 1}{3 - 1} = \frac{y - 0}{1 - 0} = \frac{z - 2}{1 - 2} = t$$

3/4/24

$x - 1 = 2t$

$x = 2t+1 \qquad dx = 2dt$

$y = t \qquad dy = dt$

$z = 2 - t \qquad dz = -dt$

$\therefore x = 2t+1$

$1 = 2t+1 \Rightarrow t = 0$

$3 = 2t+1 \Rightarrow t = 1$

$\therefore$ Line int. $= \int_{t=0}^{1} [(2t+1)^2 2dt + (2)(2t+1)(t)dt + -(2-t)-dt]$

$= \int_{t=0}^{1} [2(2t+1)^2 + 4t^2 + 2t + t - 2] dt$

$= \int_{t=0}^{1} [8t^2 + 2 + 8t + 4t^2 + 3t - 2] dt$

$= \int_{t=0}^{1} (12t^2 + 11t) dt$

$= \left[4t^3 + \frac{11t^2}{2} \right]_0^1 = 4 + \frac{11}{2} = \frac{19}{2}$

Green's Theorem (for a plane)

Consider C being the +vely oriented smooth and simple closed curve in a plane and let ~~~~ $u(x,y)$ and $v(x,y)$ and thier first partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, be a continuous and single value over the region bounded by the curve C. and $\bar{F} = u(x,y)\bar{i} + v(x,y)\bar{j}$ be any vector function in x-y plane, that ~~integral~~

$$\oint_C \bar{F} d\bar{r} = \oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy.$$

3/4/24.

where R is a closed region in the ~~x~~ XOY plane bounded by a simple closed curve C in the anti-clockwise direction.

Q. Using Green Theorem, evaluate $\int_C \bar{F}\, d\bar{r}$ for $\bar{F} = (2x^2 - 3y)i + (\tan y - e^y + 5y)j$ where C is circle $x^2 + y^2 = 4$.

Ans $\oint_C \bar{F}\, d\bar{r} = \oint_C u\, dx + v\, dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$

~~Parametric eqn of circle,~~

~~x = 2 cos t~~

By Green's Theorem,

$\oint_C \bar{F}\, d\bar{r} = \oint_C u\, dx + v\, dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$

$\oint_C u\, dx + v\, dy = \oint_C (2x^2 - 3y)\, dx + (\tan y - e^y + 5y)\, dy$

$= \oint \left(4x + \frac{1}{1-y} - e^y + 5\right)$

$\Rightarrow u = 2x^2 - 3y \qquad v = \tan y - e^y + 5y$

$\frac{\partial u}{\partial y} = -3 \qquad \frac{\partial v}{\partial x} = 0$

$\cancel{\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}} = \cancel{e^y - 0 - e^y}\ 0 - (3) = 3$

$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

3/4/24.

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy = \oint \bar{F} \cdot d\bar{r}$$

$$\oint \bar{F} \cdot d\bar{r} = \iint_R 3 \, dx\, dy$$

$$= 3 \iint_R dx\, dy$$

$x^2 + y^2 = 4$ (circle)

$= 3$ [area of circle]

$= 3 (\pi r^2)$

$= 3 (\pi)(2)^2$

$= 12\pi$

4/4/24

Q. $\bar{F} = (\sin x - 6y)\bar{i} + (\tan y)\bar{j}$ (semi-circle)

$R: y = \sqrt{1 - x^2} \Rightarrow x^2 + y^2 = 1$

By Green's theorem,

$$\oint_C \bar{F} \, d\bar{r} = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy = \oint_C u\, dx + v\, dy$$

$$\oint_C \bar{F} \, d\bar{r} = \oint_C (\sin x - 6y)\, dx + (\tan y)\, dy$$

$$= \oint_C u\, dx + v\, dy$$

$u = \sin x - 6y$, $v = \tan y$

$\frac{\partial u}{\partial y} = -6$ $\qquad \frac{\partial v}{\partial x} = 0$

$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = +6$

~~$\bar{F} = (e^{x}$ [illegible] $\tan x$ [illegible] $(e^x + 4y))\bar{i}$~~

4/4/24

$$\oint_C \bar{F}\, d\bar{r} = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy$$

$$= 6\left(\iint_R dx\, dy \right)$$

$$= 6\left(\frac{\pi r^2}{2} \right) = 3\pi r^2$$

$$= 3\pi$$

Q. Using $\bar{F} = (x^2+1)i + (y+2)j$ over the 1^{st} quad circle $x^2+y^2=4$

Ans. By Green's theorem,

$$\oint_C \bar{F}\, d\bar{r} = \oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy$$

$$u = x^2+1$$

$$v = y+2$$

$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0$$

$$\iint_R 0\, dx\, dy = 0 = \oint_C \bar{F}\, d\bar{r}$$

Date
Page

4/4/24

Q. $\bar{F} = (3x^2 - y)\bar{i} + (e^y + \log y + 4x)\bar{j}$ over the region R bounded by square whose side of length is 5 units.

Ans. By Green's theorem,

$$\oint_C \bar{F}\, d\bar{r} = \oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy.$$

$$\oint_C (3x^2 - y)\bar{i} + (e^y + \log y + 4x)\, dy$$

$$u = 3x^2 - y \qquad v = e^y + \log y + 4x$$

$$\frac{\partial u}{\partial y} = -1 \qquad \frac{\partial v}{\partial x} = 4$$

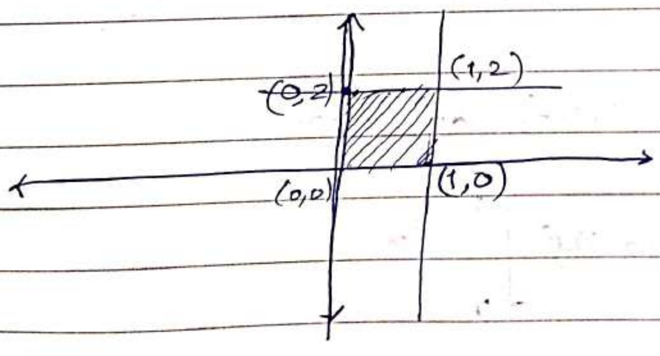
$$\oint \bar{F}\, d\bar{r} = \iint_R (4+1)\, dx\, dy$$

$$= 5 \iint_R dx\, dy$$

$$= 5(25) = 125$$

Q. Using Green's theorem, Evaluate $\oint_C \bar{F}\, d\bar{r}$ where

$\bar{F} = (\sin x + x^2 - y)\bar{i} + (\tan^{-1} y + 5x)\bar{j}$ over the region R bounded by $x = 0, x = 1, y = 0, y = 2$,



4/4/24

Ans. By Green's theorem,

$$\oint_C \bar{F}\, d\bar{r} = \oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$u = \sin x + x^2 - y$ $\quad$ $v = \tan^{-1}y + 5x$

$\frac{\partial u}{\partial y} = -1$ $\quad$ $\frac{\partial v}{\partial x} = 5$

$$\oint_C \bar{F}\, d\bar{r} = 6 \iint_R dx\,dy = 6(2) = 12.$$

Q. $\bar{F} = (\sin x + x^2 - y^2)\bar{i} + (\tan^{-1}y + 5x)\bar{j}$

$R: x = 0, x = 1; y = 0, y = 2$

By Green's theorem,

$$\oint_C \bar{F}\, d\bar{r} = \oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$\cancel{\oint_C \bar{F}\, d\bar{r}}$.

$u = \sin x + x^2 - y^2$ $\quad$ $v = \tan^{-1}y + 5x$

$\frac{\partial u}{\partial y} = -2y$ $\quad$ $\frac{\partial v}{\partial x} = 5$

$$\oint_C \bar{F}\, d\bar{r} = \iint_R 5 + 2y\; dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{2} (5 + 2y)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[5y + y^2\right]_0^2 dx$$

4/4/24.

$$= \int_{x=0}^{1} (10+4)\,dx = 14$$

Stoke's theorem

Relation between line integral and surface integral.

Surface integral of the normal component of curved of vector point funcⁿ $\bar{F}$ taken over surface S bounded by curve C = ~~surface~~ the line integral of vector point funcⁿ $\bar{F}$ taken along the closed curve C

$$\oint_C \bar{F}\, d\bar{r} = \iint_S (\nabla \times \bar{F})\, d\bar{s}$$

OR

$$\oint_C \bar{F}\, d\bar{r} = \iint_S (\nabla \times \bar{F})\, \hat{n}\, ds$$

$d\bar{s} = \hat{n} \cdot ds$ where $\hat{n}$ is a outer normal unit vector to a given open surface S bounded by simple closed curve C in the anti-clockwise direction.

Q. By using Stoke's theorem, evaluate $\oint_C \bar{F}\, d\bar{r}$ where

$\bar{F} =$ ~~$y\bar{i} + z\bar{k}$~~ $y i + z\bar{k} + x\bar{k}$

$x^2+y^2 = 1-z$, $z>0$: S

$$z = 1-(x^2+y^2)$$

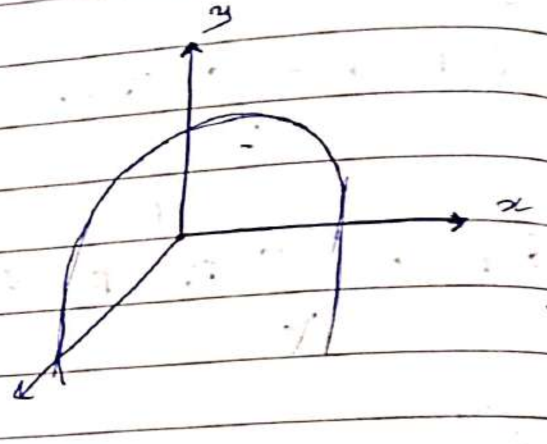
$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\, \hat{n}\cdot ds$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$\nabla \times \bar{F} = i\left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}\right) - \bar{j}\left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial z}\right) + \bar{k}\left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial y}\right)$$

$$= i(0-1) - (1-0)\bar{j} + (0-1)\bar{k}$$

$$= -\bar{i} - \bar{j} - \bar{k}$$



$S : 1 - (x^2 + y^2)$

$x = 1 - (y^2 + x^2)$

$\hat{n} = 1$

$$(\nabla \times \bar{F}) \cdot \hat{n} = (-i - \bar{j} - \bar{k})(0\bar{i} + 0\bar{j} - \bar{k})$$

$$= 1$$

$$\oint_C \bar{F}\, d\bar{r} = \iint_S (\nabla \times F) \cdot \hat{n} \; ds$$

$$= \iint_S 1 \; ds.$$

$$= \iint_{S : x^2 + y^2 = 1} ds = \qquad [ds = dx\, dy]$$

$$= \text{(area of circle)}$$

$$= \pi r^2 = \pi.$$

5/4/24

Q. $\bar{F} = 4y\,\bar{i} + 2z\,\bar{j} + 6y\,\bar{k}$

$S : z = 9 - (x^2 + y^2), \quad z > 0$

Ans. By stroke's theorem,

$\cancel{\oint \bar{F}\, d\bar{r} = \iint_S (\nabla \times F)\, d\bar{s}}$

$$\oint \bar{F}\, d\bar{s} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, d\bar{s}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 2z & 6y \end{vmatrix}$$

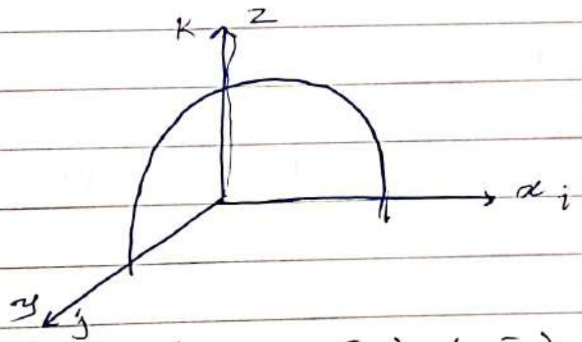
$$= \bar{i}\left(\frac{\partial(6y)}{\partial y} - \frac{\partial(2z)}{\partial z}\right) - \bar{j}\left(\frac{\partial(6y)}{\partial x} - \frac{\partial(4y)}{\partial z}\right) + \bar{k}\left(\frac{\partial(2z)}{\partial x} - \frac{\partial(4y)}{\partial y}\right)$$

$$= \bar{i}\,(6 - 2) - \bar{j}\,(0) + \bar{k}\,(0 - 4) =$$

$$= 4\bar{i} - 4\bar{k}$$

$S : z = 9 - (x^2 + y^2)$

$\hat{n} = -\bar{k}$



$$(\nabla \times \bar{F}) \cdot \hat{n} = (4\bar{i} - 4\bar{k}) \cdot (-\bar{k})$$

$$= 0 + 0 + 4$$

$$= 4$$

5/4/24

$$\oint \bar{F}\, d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$$

$$= 4 \iint_S ds$$

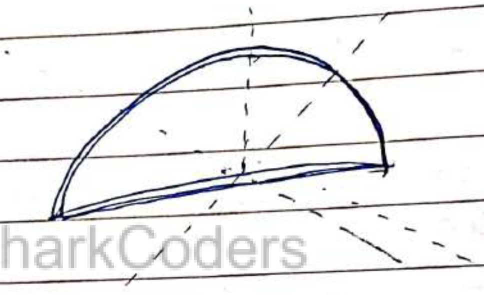
$= 4$ (area of circle $x^2 + y^2 = 9$)

$= 4\,(\pi (3)^2)$

$= 36\pi$

Q. By Stoke's theorem, evaluate $\oint \bar{F}\, d\bar{r}$ for $\bar{F} = (2x - y)\bar{i} - (yz^2)\bar{j} - (y^2 z)\bar{k}$ and S is the surf a hemisphere $x^2 + y^2 + z^2 = 1$.

Ans



By Stokes theorem

$$\oint \bar{F}\, d\bar{r} = \iint_S (\nabla \times \bar{F})\, d\bar{s} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$\cancel{\bar{i}\left(\frac{\partial(-y^2z)}{\partial y}\right.}$

$$= i\left(\frac{\partial(-y^2 z)}{\partial y} - \frac{\partial(-yz^2)}{\partial z}\right) - \bar{j}\left(\frac{\partial(-y^2 z)}{\partial x} - \frac{\partial(2x - y)}{\partial z}\right) + k\left(\frac{\partial(-yz^2)}{\partial x} - \frac{\partial(2x - y)}{\partial y}\right)$$

$= (-2yz + 2yz)i - (0)\bar{j} + (0 + \cancel{4})\bar{k}$

$= \ .y\bar{k}$

$\hat{n} = \cancel{+\bar{k}} = -\bar{k}$

$(\nabla \times \bar{F}) \cdot \hat{n} = (\bar{k}) \cdot (-\bar{k})$

$= -1.$

$$\oint \bar{F}\, d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$$

$$= -\iint_S ds$$

$= -$ (area of circle)

$= -(\pi(1)^2)$

$= -\pi$

Q. $\bar{F} = (4x + 6y)\bar{i} + (6y + 2z)\bar{j} + (2z + 4x)\bar{k}$

$S: x^2 + y^2 + z^2 = 4$ above xoz axis.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x+6y & 6y+2z & 2z+4x \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial(2z+4x)}{\partial y} - \frac{\partial(6y+2z)}{\partial z}\right) - \bar{j}\left(\frac{\partial(2z+4x)}{\partial x} - \frac{\partial(4x+6y)}{\partial z}\right) + k\left(\frac{\partial(6y+2z)}{\partial x} - \frac{\partial(4x+6y)}{\partial y}\right)$$

$= \bar{i}(0 - 2) - \bar{j}(4) + k(-6)$

$= -2\bar{i} - 4\bar{j} - 6k$

5/4/24.

$\hat{n} = -\bar{j}$

$(\nabla \times \bar{F}) \cdot \hat{n} = (-2i - 4\bar{j} - 6\bar{k}) \cdot (-j)$

$= 4$

$\oint \bar{F}\, d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$

$= 4 \iint_S ds$

$= 4$ (area of circle $x^2 + y^2 = 2^2$)

$= 4(\pi(2)^2)$

$= 16\pi$

Gauss-Divergence Theorem

Surface integral of the normal component taken over close surface S enclosing volume V = volume integral of the divergence of F taken throughout volume V.

$$\iint_S \bar{F} \cdot d\bar{s} = \iiint_V \nabla \cdot \bar{F}\, dv$$

OR

$$\iint_S \bar{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \bar{F}\, dv$$

Q. Using Gauss-divergence theorem, evaluate $\iint_S \bar{F} \cdot d\bar{s}$ where $\bar{F} = (4x + 3y^2z^2)\bar{i} + (2xz + y)\bar{j} + (y^3 +$

and S is the sphere $x + y^2 + z^2 = 16$.

Date
Page

5/4/24

Ans. By Gauss-divergence theorem,

$$\iint_S \bar{F} \cdot d\bar{s} = \iiint_V \nabla \cdot \bar{F} \, dv \quad \text{——} (1)$$

$$F_1 = 4x + 3y^2 z^2$$
$$F_2 = -2xz - y$$
$$F_3 = y^3 + 2z$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x+3y^2z^2 & -2xz-y & y^3+2z \end{vmatrix}$$

$$= i\left(\frac{\partial(y^3+2z)}{\partial y} - \frac{\partial(-2xz-y)}{\partial z}\right) - \bar{j}\left(\frac{\partial(y^3+2z)}{\partial x} - \frac{\partial(4x+3y^2z^2)}{\partial z}\right)$$
$$+ \bar{k}\left(\frac{\partial(-2xz-y)}{\partial x} - \frac{\partial(4x+3y^2z^2)}{\partial y}\right)$$

$$= (3y^2 + 2x)\bar{i} - (-6y^2z)\bar{j} + (-2z - 6yz^2)\bar{k}$$
$$= (3y^2 + 2x)\bar{i} + (6y^2z)\bar{j} - (2z + 6yz^2)\bar{k}$$

$$\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
$$= (4) + (-1) + (2)$$
$$= 5$$

$$\oint \bar{F} \, d\bar{r} = \iiint_V 5 \, dv$$

$$= 5 \,(\text{volume of sphere } - x^2+y^2+z^2 = 16)$$

$$= 5\left(\frac{4\pi r^3}{3}\right) = \frac{20\pi r^3}{3}$$

Q. $\bar{F} = (6x + yz)\bar{i} + (\sin \cancel{xy} z + y)\bar{j} + (e^{xy} + z)\bar{k}$

$S : x^2 + y^2 + z^2 = 9.$

Ans. $\iint \bar{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv$ ——— (1)

$F_1 = 6x + yz$ $\quad F_2 = \sin xz + y$ $\quad F_3 = e^{xy} - z$

$$\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

$= 6 + 1 - 1$

$= 6$

$$\iint_S \bar{F} \cdot \hat{n} \, d\bar{s} = \iiint_V \nabla \cdot \bar{F} \, dv$$

$$= 6 \iiint_V dv$$

$$= 6 \iiint_V dx \, dy \, dz$$

$$= 6 \left(\frac{4}{3} \pi r^3 \right) = \cancel{72\pi} \; 216\pi$$

Q. ~~[illegible]~~ $S : x^2 + y^2 + z^2 = 4$

$\bar{F} = 3x\bar{i} - 2y\bar{j} + 2z\bar{k}$

Ans. $\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$= 3 - 2 + 2$

$= 3$

$$\iint_S (\nabla \times \bar{F}) \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv$$

$$= 3 \iiint_v dv$$

$$= 3 \iiint_V dx\, dy\, dz$$

$= 3$ (~~the~~ volume of sphere)

$$= 3\left(\frac{4\pi r^3}{3}\right) = 32\pi$$

Q. $\bar{F} = (2x + yz)\bar{i} + (3y + xz)\bar{j} + (-z + e^{x^3})\bar{k}$

S : cube with side 5 unit.

Ans. $\nabla \cdot \bar{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

$$= 2 + 3 - 1$$

$$= 4$$

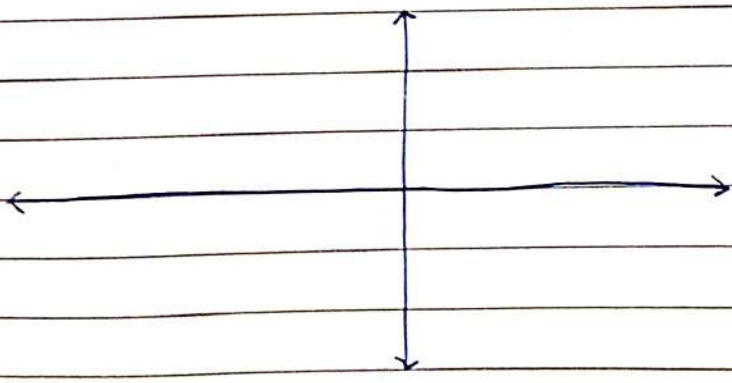
$$\oint \bar{F}\, d\bar{r} = \iiint_V (\nabla \cdot \bar{F})\, dv$$

$$= 4 \iiint_V dx\, dy\, dz$$

$$= 4\,(5^3) = 500$$

Q. $\bar{F} = 5x\,\bar{i} + 6y\,\bar{j} - 3z\,\bar{k}$

S : $x = 0,\ x = 2,\ y = 0,\ y = 2,\ z = 0,\ z = 3.$



$$\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 8$$

$$\oint \bar{F}\, d\bar{r} = \iiint_V \nabla \cdot \bar{F}\, dv$$

$$= 8 \iiint_{xyz} dx\, dy\, dz$$

$$= 8 \int_0^2 dx \cdot \int_0^2 dy \int_0^3 dz$$

$$= 8\, [x]_0^2\, [y]_0^2\, [z]_0^3$$

$$= 8\,(2)(2)(3)$$

$$= 96$$

16/4/24

Unit VI: Numerical Methods

Interpolation

Suppose the following values are given. ~~of y=f(x)~~ $y = f(x)$ for a set of values x

x	x_0	x_1	x_2	...	x_n
$y=f(x)$	y_0	y_1	y_2	...	y_n

The process of finding the values of y corresponding to any value of $x = x_i$ between x_0 to x_n is interpolation.

Therefore, interpolation is a technique of estimating the value of function for any intermediate value of the independent variable.

Lagrange's Interpolation formula

There are many interpolation formulae that are available only for equally spaced values of the arguement x.

This formula is for unequally spaced arguement values. If $y=f(x)$, take the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x_0, x_1, x_2, \ldots, x_n$. Then the polynomial passing through the given points is given by

$$y = P_n(x) = \sum_{i=0}^{n} L_i(x)\, y_i \quad \cancel{\text{where } L_i(x) = (x-x_0)(x-x_1)\ldots}$$

where

$$L_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

$$y = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 + \dots + L_n(x)y_n$$

$$= \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1 + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

- can also be used for equally spaced values of x.

Q1. Find Lagrange's interpolating ~~formula~~ polynomial passing through set of points ~~xxx~~

x	0	1	2
y	4	3	6

Use it to find y at $x = 1.5$, $\frac{dy}{dx}$ at $x = 0.5$ and

$$\int_0^3 y\,dx$$

Ans. $y = P_n(x) = \sum_{i=0}^{n} L_i(x) \cdot y_i$

$x_0 = 0 \qquad y_0 = 4$

$x_1 = 1 \qquad y_1 = 3$

$x_2 = 2 \qquad y_2 = 6$

$$y = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)}(4) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(3) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(6)$$

$$= \frac{4(x-1)(x-2)}{2} + \frac{3(x)(x-2)}{-1} + \frac{6(x)(x-1)}{2}$$

$$= 2(x-1)(x-2) - 3x(x-2) + 3x(x-1)$$

16/4/24

classmate

Date

Page

$= \cancel{2(x^2 - 3x + 2) - 3}$

$= 2(x^2 - 3x + 2) - 3(x^2 - 2x) + 3(x^2 - x)$

$= 2x^2 - 6x + 4 - 3x^2 + 6x + 3x^2 - 3x$

$= 2x^2 - 3x + 4$

Q2. $[y_P]_{x=1.5} = [2x^2 - 3x + 4]$

$= 2(1.5)^2 - 3(1.5)^2$

$= 4$

$y = 2x^2 - 3x$

$$\int_0^3 y\,dx = \int_0^3 (2x^2 - 3x + 4)\,dx$$

$$= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 4x\right]_0^3 = 16.5$$

Q. Find Lagrange's interpolating polynomial passing through a set of points (0,2), (2,-2), (3,-1). Use it to find y at $x=1$, $\frac{dy}{dx}$ at $x=2$, $\int_0^1 y\,dx$.

$x_0 = 0$ $\quad y_0 = 2$

$x_1 = 2$ $\quad y_1 = -2$

$x_2 = 3$ $\quad y_2 = -1$

$$y = \sum_{i=0}^{n} L_i(x)\, y_i$$

$$= L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$= \frac{(x-2)(x-3)}{(0-2)(0-3)}(2) + \frac{(x-0)(x-3)}{(2-0)(2-3)}(-2) + \frac{(x-0)(x-2)}{(3-0)(3-2)}(-1)$$

16/4/24

$$= \frac{2(x-2)(x-3)}{6} + \frac{2x(x-3)}{2} + \frac{x(x-2)}{3}$$

$$= \frac{(x-2)(x-3)}{3} + (x^2-3x) - \frac{(x^2-2x)}{3}$$

$$= \frac{x^2-3x-2x+6+3x^2-9x-x^2+2x}{3}$$

$$= \frac{3x^2-12x+6}{3} = x^2-4x+2.$$

y at $x=1$

$y = (1)^2 - 4(1) + 2$

$= 1 - 4 + 2$

$y = -1$

$$\frac{dy}{dx} = \frac{d(x^2-4x+2)}{dx} = 2x-4 \text{ at } x=2$$

$$\therefore \frac{dy}{dx} = 0$$

$$\int_0^3 y\,dx = \int_0^3 (x^2-4x+2)\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 2x\right]_0^1$$

$$= \cancel{9 - 18 + 6 = -3}$$

$$= \left(\frac{1}{3} - 2 + 2\right) = \frac{1}{3}$$

16/4/24

Q. Find LIP passing through set of points

x	0	1	3
y	3	4	12

Use it to find y at $x = 0.5$ and $\frac{dy}{dx}$ at $x = 1$.

Ans. $x_0 = 0$ $\quad y_0 = 3$

$x_1 = 1$ $\quad y_1 = 4$

$x_2 = 3$ $\quad y_2 = 12$.

$$y = P_n(x) = \sum_{i=0}^{n} L_i(x) y_i$$

~~$y = (x - x_1)$~~

$$y = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2.$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)}(3) + \frac{(x-0)(x-3)(4)}{(1-0)(1-3)} + \frac{(x-0)(x-1)(12)}{(3-0)(3-1)}$$

$$= \frac{3(x-1)(x-3)}{3} + \frac{4(x)(x-3)}{-2} + \frac{12(x)(x-1)}{6}$$

$$= x^2 - 3x - x + 3 - 2x(x-3) + 2x(x-1)$$

$$= \cancel{x^2} \cancel{-4x} + 3 \cancel{-2x^2} \cancel{+6x} \cancel{+2x^2} \cancel{-2x}$$

$$= x^2 + 3$$

y at $x = 0.5$ $\qquad \frac{dy}{dx} = 2x = 2(1) = 2.$

$y = x^2 + 3$

$= (0.5)^2 + 3$

$= 3.25$

SharkCoders

19/4/24

Finite Difference and Difference Operator

1) Forward operator (Δ - del)

$$\Delta f(x) = f(x+h) - f(x)$$

The difference $\nabla f(x)$ being the difference between the value of the function at forward point current point.
For x and y with a difference of h between consecutive x values.

x	x_0	x_1	x_2	x_3	x_4
y	y_0	y_1	y_2	y_3	y_4

$$\Delta y_0 = y_1 - y_0$$

~~$\Rightarrow \Delta^2 y_0 = \Delta(\Delta y_0)$~~

forward ~~$= \Delta(y_1 - y_2)$~~

~~Second~~ Difference ~~operator~~ of second order :

$$\Delta^2 y_0 = \Delta(\Delta y_0)$$
$$= \Delta(y_1 - y_0)$$
$$= \Delta y_1 - \Delta y_0$$
$$= (y_2 - y_1) - (y_1 - y_0)$$
$$= y_2 - y_1 - y_1 + y_0$$
$$\Delta^2 y_0 = y_0 - 2y_1 + 2y_2$$

19/4/24

2) Backward Operator (∇ - nebla)

The difference $\nabla f(x)$ being the difference between value of function at current point at the receding point. is backward difference of first order. It is defined as:

$$\nabla f(x) = f(x) - f(x-h)$$

OR

$$\nabla y_1 = y_1 - y_0$$

3) Shift Operator (E)

$$Ef(x) = f(x+h) \quad \text{[First order]}$$

$$Ef(x) = f(x+2h) \quad \text{[Second order]}$$

$$E^r f(x) = f(x+rh)$$

$$E^{-r} f(x) = f(x-rh)$$

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_0+h	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_0+2h	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_0+3h	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
		$\Delta y_3 = y_4 - y_3$			
x_0+4h	y_4				

19/4/24.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
$x_0 + h$	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
$x_0 + 2h$	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$
$x_0 + 3h$	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$	
		$\nabla y_4 = y_4 - y_3$		
$x_0 + 4h$	y_4			

$\nabla^4 y_4$

<u>Newton forward Difference Interpolation Formula</u> (NFDIF)

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \dots$$

where $u = \dfrac{x - x_0}{h}$

<u>Newton Backward Difference Interpolation formula</u> (NBDIF)

$$y = y_n + \frac{u\nabla y_n}{1!} + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n$$

where $u = \dfrac{x - x_n}{h}$

<u>NFDIF</u> ~~is used to estimate~~

- used to estimate y for the value of x which a[re] very close to x_0 i.e. at the beginning of the table

<u>NBDIF</u>

- used to estimate y for the value of x which ar[e] very close to x_n i.e. at the end of the table.

19/4/24

Q. Given that

x	4	6	8	10
y	1	3	8	16

Using NIF, find the value of y when $x = 5$.

Ans. $x = 5$ (closer to x_0)

$x_0 = 4$

$h = 2$

$$u = \frac{x - x_0}{h} = \frac{5-4}{2} = \frac{1}{2}$$

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1 Δy_0			
		2 $\Delta^2 y_0$		
			3 $\Delta^2 y_0$	
6	3			0 $\Delta^3 y_0$
		5		
8	8		3	
		8		
10	16			

$$y = 1 + \left(\frac{1}{2}\right)(2) + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)(3) + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{1}{6}\right)0 =$$

$$= 1 + 1 - \frac{3}{8} \quad \text{at } x = 5$$

$$= \frac{13}{8} = 1.625 \quad \text{at } x = 5.$$

19/4/24.

Q. Using NFF, find cubic polynomial which takes the following values.

x	0	1	2	3
y	1	2	1	10

$x = x$

$x_0 = 0$

$h = 1$

$u = x$

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 0 + x(1) + \frac{2x(x+2)}{2}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

$$y = 1 + x(1) + \frac{x(-x+1)(-2)}{2} + \frac{2x(x-1)(x-2)(12)}{6}$$

$$= 1 + x + x - x^2 + 2x(x^2 - 3x + 2)$$

$$= 1 + 2x - x^2 + 2x^3 - 6x^2 + 4x$$

$$= 1 + 6x - 7x^2 + 2x^3$$

$$= 2x^3 - 7x^2 + 6x + 1$$

19/4/24

Given that

x	0	1	2	3
y	1	2	1	10

Using NIF, find the value of y ~~at~~ where $x = 4$.

Ans.

$x = 4$

$x_n = 3$

$h = 1$

$$u = \frac{x - x_n}{h} = \frac{4-3}{1} = 1.$$

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \dots$$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	1			
		+1		
1	2		-2	
		-1		12
2	1		10	
		+9		
3	10			

$$\cancel{y = (10) + (1)(9) + (1)(0) + (0)}$$

$$\cancel{= 10 + 10 = 20}$$

$$y = (10) + (1)(9) + \frac{(1)(2)}{2}(10) + \frac{(1)(2)(3)}{6}(12)$$

$$= 10 + \cancel{10}\,9 + 10 + 12$$

$$= \cancel{40}\ 41 \quad \text{at } x = 4$$

19/4/24

Q. The distance travelled by a point P in x-y plane in a mechanism is shown in the table below. Estimate the distance travelled, velocity and acceleration of a point P when $x = 4.5$.

x	1	2	3	4	5
y	14	30	62	116	198

$x = 4.5$ $\qquad x = x$

$x_n = 5$ $\qquad x_n = 5$

$h = 1$ $\qquad h = 1$

$u = \dfrac{x - x_n}{h}$ $\qquad u = \dfrac{x - x_n}{1} = (x-5)$

$= \dfrac{4.5 - 5}{1} = \dfrac{-1}{2}$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	14				
		16			
2	30		16		
		32		6	
3	62		22		0
		54		6	
4	116		28		
		82			
5	198				

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$= (198) + (x-5)(82) + \frac{(x-5)(x-4)}{2}(\cancel{28})^{14} + \frac{(x-5)(x-4)(x-3)(6)}{6} + 0$$

19/4/24.

$= 198 + 82x - 410 - 14(x^2 - 4x - 5x + 20) + (x-3)(x^2 - 4x - 5x + 20)$

$= 198 + 82x - 410 - 14x^2 + 126x - 280 + (x^3 - 9x^2 + 20x - 3x^2 + 27x - 60)$

$= \cancel{198} + 82x - 410 - \cancel{14x^2} + 126x - 280 + x^3 - 12x^2 + 47x - 60$

$= x^3 - 26x^2 + 255x - 552$

$y = x^3 - 26x^2 + 255x - 552.$

$\frac{dy}{dx} = .$

$= 198 + 82x - 410 + 14(x^2 - 4x - 5x + 20) + (x-3)(x^2 - 9x + 20)$

$= 198 + 82x - 410 + 14x^2 - 56x - 70x + 280 + (x^3 - 9x^2 + 20x - 3x^2 + 27x - 60)$

$= \cancel{198} + \cancel{82x} - \cancel{410} + \cancel{14x^2} - \cancel{126x} + \cancel{280} + x^3 - \cancel{12x^2} + \cancel{47x} - 60$

$= x^3 + 2x^2 + 3x + 8 = y$

$\frac{dy}{dx} = 3x^2 + 4x + 3 = v$

$\frac{d^2y}{dx^2} = 6x + 4 = a$

y at $x = 4.5$

$y = 153.125$

$v = 81.75$

$a = 31$

Interpolation

- Unequally spaced
  - Lagrange's interpolation formula

    $y = P_n(x) = \sum_{i=0}^{n} L_i(x) y_i$

    where $L_i(x) =$
- Equally spaced
  - NFDIF — close to x_0

    $y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!} y_0 \Delta^2 + \dots$

    where $u = \frac{x - x_0}{h}$
  - NBDIF — close to x_n

    $y = y_n + u\nabla y_n + \frac{u(u-1)}{2!} \dots$

    where $u = \frac{x - x_n}{h}$

Q. Apply

19/4/24

Numerical Solutions of Algebraic and Transcdental Eq^n

An eq^n $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$ $[a_n \neq 0]$ is called an algebraic eq^n.

Eg.: 1) $5 + 4x + 6x^2 + 7x^4 = 0$ 2) $2x - 4x^3 + 7x^9 = 0$

Any non-algebraic eq^n is called a transcdental eq^n.

Eg.:

OR

An eq^n containing trignometric, exponential, logarithmic, ~~funct~~ etc. function is called transcdental eq^n

Eg.: 1) $x\cos x - 1 = 0$ 2) $\sin x + \cos x = 0$

3) $xe^x - 5 = 0$ 4) $\log x + 5\tan x - 1 = 0$

Theorem

If, for $f(x) = 0$, we can find two values a and b such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has at least one real root between a and b.

Eg.: $x^2 + 2x - 1 = 0$

$f(0) = -1 < 0$

$f(1) = 2 > 0$

Numerical to solve algebraic/transcdental eq^n

1. Bisectioning method.

Let $f(x) = 0$ be a given eq^n to be solved using this method

I. To find the values of a and b of x such that $f(a) < 0$ and $f(b) > 0$ (can be vice versa).

II. Hence $f(x) = 0$ has at least one ~~root~~ real root between a and b.

18/4/24

III. Let its first approximation be $\left(x_1 = \frac{a+b}{2}\right)$.

IV. Find $f(x_1)$.

V. If $f(x_1) < 0$, $f(x) = 0$ has at least one real root between x_1 to b

If $f(x_1) > 0$, $f(x) = 0$ has at least one real root between a to x_1.

VI. If $f(x_1) < 0$, $[x_1 \text{ to } b]$

second approximation $x_2 = \frac{x_1 + b}{2}$

find $f(x_2)$

If $f(x_1) > 0$, $[a \text{ to } x_1]$

second approximation $x_2 = \frac{x_1 + a}{2}$

find $f(x_2)$

VII. Repeat to get approximation of roots.

NOTE: THIS METHOD IS VERY SLOW

Q1. Apply bisection method to find the root of the eq. $x^3 - 9x + 1 = 0$ correct to 3 decimal places.

Ans. $f(x) = x^3 - 9x + 1$

$f(0) = 0 - 0 + 1 = 1 > 0$

$f(1) = 1 - 9 + 1 = -7 < 0$

First app.,

$a = 0 \qquad \frac{a+b}{2} = \frac{0+1}{2} = 0.5$

$b = +1$

Second app.,

~~a = +0.5~~

$f(0.5) = -3.375 < 0$

18/4/24

Second app.,

0.5 to 0

$a = 0.5$

$b = 0$

$$\frac{a+b}{2} = \frac{0.5}{2} = 0.25$$

$f(0.25) = -1.234$

Third app.,

$a = 0.25$

$b = 0$

$$\frac{a+b}{2} = \frac{0.25}{2} = 0.125$$

$f(0.125) = -0.123$

Fourth app.,

$a = 0.125$

$b = 0$

$$\frac{a+b}{2} = \frac{0.125}{2} = 0.0625$$

$f(0.0625) = 0.4377$

Fifth app.,

$a = 0.125$

$b = 0.0625$

$$\frac{a+b}{2} = \frac{0.125+0.0625}{2} = \cancel{0.0934} \; 0.09375$$

~~f(0.094) = 0.155 > 0~~

$f(0.09375) = 0.1570 > 0$

Sixth app.,

$a = 0.125$

$b = 0.09375$

$$\frac{a+b}{2} = 0.109375$$

$f(0.109375) = 0.01693$

18/4/24

Seventh app.,

$a = 0.125$ $\qquad \frac{a+b}{2} = \frac{0.125 + 0.109375}{2} = 0.1172$

$b = 0.109375$

$f(0.1172) = -0.05$

Eigth app.,

$a = 0.1172$ $\qquad \frac{a+b}{2} = \frac{0.1172 + 0.109375}{2} = 0.113$

$b = 0.109375$

0.11 is the root.

Q. Apply bisection method to solve $x^3 - x - 1 = 0$ correct upto 3 decimal places.

Ans. $\because f(x) = x^3 - x - 1$

$f(0) = 0 - 0 - 1 = -1 < 0$

$f(1) = 1 - 1 - 1 = -1 < 0$

$f(2) = 8 - 2 - 1 = 5 > 0$

$a = 1$ $\qquad \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5$

$b = 2$

First app., $f(1.5) = 0.875 > 0$

Second app.,

$a = 1$ $\qquad \frac{a+b}{2} = \frac{2.5}{2} = 1.25$

$b = 1.5$

$f(1.25) = -0.2968 < 0$

Third app.,

$a = 1.25$ $\qquad \frac{a+b}{2} = \frac{2.75}{2} = 1.375$

$b = 1.5$

$f(1.375) = 0.2246 > 0$

Fourth app.,

$a = 1.25$ $\qquad \frac{a+b}{2} = \frac{2.625}{2} = 1.3125$

$b = 1.375$

$f(1.3125) = -0.05151 < 0$

Fifth app.,

$a = 1.3125$ $\qquad \frac{a+b}{2} = \frac{2.6875}{2} = 1.34375$

$b = 1.375$

$f(1.34375) = 0.08261 > 0$

Sixth app.,

$a = 1.3125$ $\qquad \frac{a+b}{2} = \frac{2.65625}{2} = 1.328125$

$b = 1.34375$

$f(1.3281) = 0.01447 > 0$

Seventh app.,

$a = 1.3125$ $\qquad = \frac{a+b}{2} = \frac{2.6406}{2} = 1.3203$

$b = 1.3281$

$f(1.3203) = -0.0187$

$\therefore$ 1.32 is the root.

23/4/24.

2. Secant Method.

Let $f(x)=0$ be a given eqn to be solved using method of secant method with initial approximations say, x_0 and x_1.

1) To get approximation use
$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad ; n = 1, 2, 3, \ldots$$

2) Repeat above procedure to get best approxi.. of roots

Q. Use secant method to find the root of the eq $f(x) = x^3 - x - 1$ to 3 decimal places.

Ans $f(x) = x^3 - x - 1$

$f(0) = -1$

$f(1) = -1 < 0$

$f(2) = 5 > 0$

$x_0 = 1$

$x_1 = 2$

~~First app.~~ By secant method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad ; n = 1, 2, 3, \ldots$$

First app., $(n=1)$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{1(5) - 2(-1)}{5 + 1} = \frac{7}{6}$$

23/4/24

$x_2 = \cancel{1.6}\ \cancel{1.0667}\ 1.1667$

Second app., $(n=2)$

$\cancel{x}$.

$x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$ $\qquad f(\overset{x_2}{1.1667}) = -0.5786$

$x_3 = \dfrac{2(-0.5786) - (1.1667)(5)}{(-0.5787) - 5}$

$x_3 = 1.2531$

Third, app., $(n=3)$

$\cancel{x}$

$x_4 = \dfrac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$ $\qquad f(\overset{x_3}{1.2531}) = -0.2854$

$= \dfrac{1.1667(-0.2854) - 1.2531(-0.5786)}{(-0.2854) - (-0.5786)}$

$= 1.337$

Fourth app., $(n=4)$

$x_5 = \dfrac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$ $\qquad f(\overset{x_4}{\overset{1.337}{\cancel{0.0530}}}) = 0.0530$

$x_5 = \dfrac{(1.2531)(0.0529) - (1.337)(-0.2854)}{(0.0529) - (-0.2854)}$

$= 1.3238$

Fifth app., $(n=5)$

$x_6 = \dfrac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)}$

$= \dfrac{(1.337)(-0.0039) - (1.3238)(0.0530)}{\cancel{0.0530 - (-0.2854)}\ (-0.0039) - (0.0529)}$

$= \cancel{1.3338}\ 1.3247.$

23/4/24

Sixth app., $(n=6)$

$$x_7 = \frac{x_5 f(x_6) - x_6 f(x_5)}{f(x_6) - f(x_5)}$$

$$= \frac{(1.3232)(0.00007) - (1.3247)(-0.0029)}{(-0.00007) - (-0.0029)}$$

$$= 1.3247$$

$\therefore$ 1.3247 is a real root correct upto 4 decimal places.

Q. Use secant method to find the root of the func $f(x) = x^3 - 4$, to 3 decimal places (Given $x_0 = 1$; $x_1 = 1.5$)

Ans. $f(x_1) = -0.625$ $\quad f(x_0) = -3$

First approximation $(n=1)$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1(-0.625) - (1.5)(-3)}{-0.625 + 3}$$

$$= 1.63105$$

x_2

$f(1.6315) = 0.3427$

Second app., $(n=2)$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{1.5(0.3427) - 1.6315(-0.625)}{(0.3427) - (-0.625)}$$

$$= \cancel{1.585} \; 1.5849$$

x_3

$f(1.5849) = -0.011$

...

23/4/24

Third appx $(n=3)$;

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \qquad f(x_4) = -0.0007$$

$$= \frac{(1.6215)(-0.0188) - (1.5849)(0.3427)}{(-0.0188) - (0.3427)}$$

$$= 1.5873$$

Fourth appx $(n=4)$

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$= \frac{(1.5849)(-0.0007) - (1.5873)(-0.0188)}{(-0.0007) - (-0.0188)}$$

$$= 1.5873$$

Hence, 1.5873 is a real root correct upto 4 decimal places.

Q. Using secant method, to find the root of the function $f(x) = x^3 - 9x + 1$. (~~Given x~~

Ans. $f(2) = -9$

$f(3) = 1$

3) ~~Ned~~ Newton - Ruphson Method.

~~$f(x)=0$~~ When $f(x)=0$, be ~~the~~ a given eq^n to be solved ~~during~~ using newton - ruphson method.

1) Find initial approximation of $f(x)=0$, say x_0.

2) To get approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; n = 0, 1, 2, 3, \dots$$

3) Repeat above procedure to get best approximation of roots.

~~Remarks:~~

Ex 1. Use newton-ruphson method to find root of eq^n $x^3 - 5x + 3 = 0$ correct to 3 decimal places.

Ans. Let $f(x) = x^3 - 5x + 3$

$f'(x) = 3x^2 - 5$

$f(0) = 3 > 0$

$f(1) = -1 < 0$

$\therefore$ By Newton Ruphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x)_n}$$

0^{th} ~~first~~ app., $(n=0)$

Let $x_0 = 0.5 \in [0, 1]$

First app. $(n=0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{0.625}{-4.25}$$

$$= 0.647$$

Second app. $(n=1)$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.647 - \frac{0.0358\ldots}{-3.7442\ldots}$$

$$= 0.65605$$

Third app., $(n=2)$

$$x_3 = 0.6565 - \frac{0.00045}{-3.7070}$$

$$= 0.6566.$$

Hence 0.6566 is the real root correct upto 4 decimal places.

Q. Use newton-raphson method to find root of eqn $x^3 - x - 1 = 0$ correct upto 3 decimal places.

Ans. Let $f(x) = x^3 - x - 1 = 0$

$f'(x) = 3x^2 - 1$

$f(1) = -1 < 0$

$f(2) = 5 > 0$

Let $x_0 = 1.7$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

First app., $(n=0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.7 - \frac{2.213}{7.67} = 1.4115$$

Second app., $(n=1)$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.4115 - \frac{f(1.4115)}{f'(1.4115)}$$

$$= \cancel{1.330}\ 1.331$$

Third app., $(n=2)$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.32487$$

Fourth app., $(n=3)$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.3247$$

Hence 1.3247 is a real root correct upto 4 decimal places

Q. Find the approximate value of $\sqrt{28}$ correct to 3 decimal places using newton - raphson method

Ans. $\sqrt{25} = 5$

$\sqrt{36} = 6$

Let $x = \sqrt{28}$

$x^2 = 28$

$f(x) = x^2 - 28 = 0$

$f'(x) =$

$f(0) = -28$

$f(1) = -27$ $\quad \cancel{f(2) = 2}$ $f(1$

$f(5) = 25 - 28 = 3 < 0$

$f(6) = 36 - 28 = 8 > 0$

$x_0 = 5.5$

$f(x) = x^2 - 28$

$f'(x) = 2x$

First app., $(n=0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 5.5 - \frac{2.25}{11}$$

$$= 5.2954$$

Second app. $(n=1)$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 5.2915$$

Third app. $(n=2)$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= \cancel{7.2553}\ 5.2915$$

~~Fourth app., (n=3)~~

$$\cancel{x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}}$$

Hence, 5.2915 is a real root correct upto 4 decimal places.

Q. Find the approximate value of $\sqrt[3]{13}$ correct to 3 decimal places using newton-raphson method.

Ans. $\cancel{2\sqrt{13} = \sqrt{52} = x}$

$\cancel{f(x) = x^2 - 52 = 0}$

$\cancel{f'}$

$\sqrt[3]{13} = x$

$x^3 = 13.$

$f(x) = x^3 - 13 = 0.$

$f(2) = x^3 - 13$ 　　　 $f(x) = x^3 - 13$

$f(2) = -5 < 0$ 　　　 $f'(x) = 3x^2$

$f(3) = 14 > 0$

$x_0 = 2.5$

First app., $(n=0)$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 2.36$

Second app. $(n=1)$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= \cancel{2.3396}\ 2.35139$

Third app. $(n=2)$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= \cancel{2.3671}\ 2.3513$

~~Fourth app., (n=3)~~

$\cancel{x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}}$

$=$

Hence, 2.3513 is a real root correct upto 4 decimal places.

Q. $\sqrt[4]{\frac{1}{3}}$.

Ans. $\sqrt[4]{\frac{1}{3}} = x$

$x^4 - \frac{1}{3} = 0 = f(x)$

$f(0) = \cancel{\frac{-1}{3}}\ \frac{-1}{3}$

$f(1) = \frac{2}{3}$

$x_0 = 0.5$

$f(x) = x^4 - \frac{1}{3}$.

$f'(x) = 4x^3$

First app. $(n=0)$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = \cancel{3.3114}\ 1.0417$

Second app. $(n=1)$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.0417$

Hence, 1.0417 is a real root when

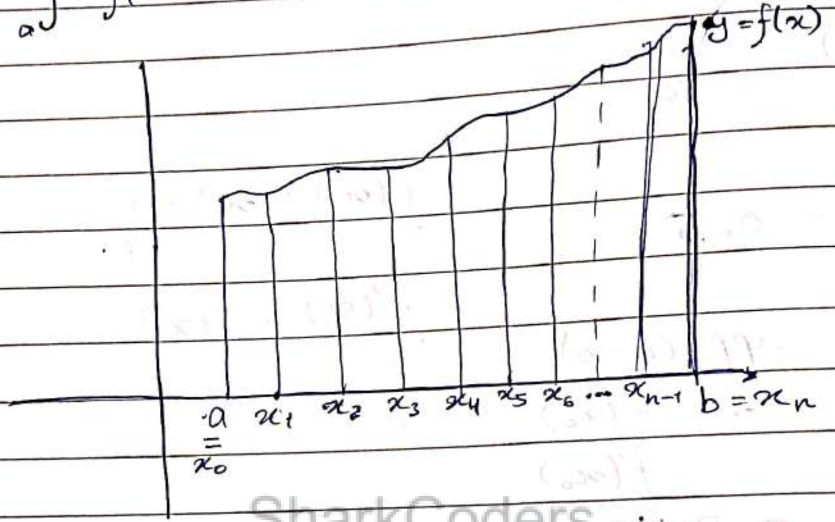
25/4/24

Unit VI : Numerical Method - II

Numerical Integration

The area bounded by the curve $y = f(x)$ and y-axis between limit a and b denoted by

$$I = \int_a^b f(x)\,dx.$$



Now we divide (a, b) into n-equal interval (parts) with length h (step size)

$$[a, b] = [a = x_0, x_1, x_2, \ldots, x_n = b]$$

$a = x_0$ $\qquad$ $h = \dfrac{b-a}{n}$ OR $n = \dfrac{b-a}{h}$

$x_1 = x_0 + h$

$x_2 = x_1 + h$

$x_3 = x_2 + h$

$\vdots$

$x_n = x_{n-1} + h$

The process of evaluating an ~~ite~~ definite integral

$I = \int_a^b f(x)\,dx$ from a set of tabulated values of $f(x)$ is called numerical integration.

Rules to find Numerical Int.

1. Trapezoidal Rule

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + (2)(y_1+y_2+y_3+\dots+y_{n-1})\right]$$

2. Simpson's $(1/3)^{rd}$ Rule

$$I = \int_a^b f(x)\,dx =$$

$$= \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\dots+y_{n-1}) + 2(y_2+y_4+y_6+\dots+y_{n-2})\right]$$

3. Simpson's $(3/8)^{th}$ Rule

$$I = \int_a^b f(x)\,dx$$

$$= \frac{3h}{8}\left[(y_0+y_n) + (2)(y_3+y_6+y_9+\dots) + 3(y_1+y_2+y_4+y_5+\dots)\right]$$

Notes

1. There is no restriction for the number of intervals in a trapezoidal rule.

2. In Simpson's $1/3^{rd}$ Rule, the no. of sub-interval rule, the no. of sub-interval must be even.

3. In Simpson's 3/8 Rule, the no. of sub-interval must be multiple of 3.

4. In Simpson's 3/8 Rule, to get more accuracy.

25/4/24

- divide the given interval into maximum no. of sub-interval.

5. no. of ordinate = (no. of subinterval) + (1)

Q. $I = \int_0^4 y \, dx$; $n = 8$

$$h = \frac{b-a}{n} = \frac{4-0}{8} = \frac{1}{2}.$$

1	2	3	4	5	6	7	8	9	= 8+1.
0	1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2	

Q. Evaluate $I = \int_0^1 \frac{1}{1+x} dx$ dividing the range into 6 equal parts.

Ans. $n = 6$

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6} \quad ; n = 6$$

$$y = f(x) = \frac{1}{1+x}, \quad a = 0 \; ; \; b = 1, \; n = 6$$

x	0	1/6	2/6	3/6	4/6	5/6	6/6
$y = \frac{1}{1+x}$	1	0.857	0.75	0.67	0.6	0.54	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Trapezoidal Rule :

$$I = \int_0^1 \frac{1}{1+x} dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

25/4/24

$$I = \int_0^1 \frac{1}{1+x}\,dx$$

$$I = \frac{1/6}{2}\left[(1+0.5) + 2(0.857+0.75+0.67+0.6+0.54+0.5)\right]$$

$$= 0.7779$$

By Simpson's 1/3-rd Rule,

$$I = \int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_6) + 4(0.857+0.67+0.54) + 2(0.75+0.6+0.5)\right]$$

$$= 0.7482 = 0.6931$$

By Simpson's 3/8th Rule,

$$I = \int_a^b f(x)\,dx = \frac{3h}{8}$$

$$= \frac{3(1/6)}{8}\left[(1.5) + 2(0.6667+0.5) + 3(0.857+0.75+0.6+0.5454)\right]$$

$$= \cancel{0.7850} \; 0.6932$$

Que $I = \int_0^1 \frac{1}{1+x}\,dx.$

$$= [\log(1+x)]_0^1$$

$$= \log 2 - \log 1$$

$$= \log 2 - 0$$

$$= 0.6931$$